Nonlinear Fokker-Planck equations with Fractional Laplacian and McKean-Vlasov SDEs with Lévy-Noise

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Reference:
Barbu/R.: AOP 2020
Barbu/R.: JFA 2023
Barbu/R.: arXiv*: 2210.05612
Rehmeier/R.: arXiv: 2212.12424
Ren/R./Wang: JDE 2022
R./Xie/Zhang: PTRF 2020

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0. Motivation and longterm programme: Recall Classical Case (Linear!)

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ANALYSIS

Heat equation on $\mathbb{R}^d$:

$$\frac{\partial}{\partial t} u(t,x,y) = \frac{1}{2}\Delta_y u(t,x,y),\ (t,y) \in (0,\infty) \times \mathbb{R}^d$$
$$u(0,x,y) = \delta_x(y) \ (= \text{Dirac measure in } x \in \mathbb{R}^d)$$

Solution: Classical **heat kernel**

$$u(t,x,y) = \frac{1}{(2\pi t)^{\frac{d}{2}}}\, e^{-\frac{1}{2t}\|y-x\|^2}$$

PROBABILITY

Wiener measure $\mathbb{W}_x$ on $C([0,\infty);\mathbb{R}^d)_x$ [Wiener 1920]
For $W(t) : C([0,\infty);\mathbb{R}^d)_x \to \mathbb{R}^d$,
$W(t)(w) := w(t),\ t \geq 0$,

$$\underbrace{(W(t))^*(\mathbb{W}_x)}_{\text{"push forward"}}(\mathrm{d}y) = u(t,x,y)\mathrm{d}y,\ t > 0$$

$(W(t))_{t\geq 0}, \mathbb{W}_x)_{x\in\mathbb{R}^d}$ "Wiener process"

(="Brownian motion")

GENERAL

Linear Parabolic PDE (more precisely: **linear** Fokker-Planck equation)

↕

linear Markov process (described by SDE)

0. Motivation and longterm programme: Nonlinear Case (Projects A5 and B1, CRC 1283)

EXAMPLE

A N A L Y S I S

Porous media equation on $\mathbb{R}^d$: For $m \geq 1$

$$\frac{\partial}{\partial t} u(t,x,y) = \frac{1}{2}\Delta_y(|u|^{m-1}u)(t,x,y),\ (t,x) \in (0,\infty) \times \mathbb{R}^d$$

$$u(0,x,y) = \delta_x(y)\ (= \text{Dirac measure in } x \in \mathbb{R}^d)$$

Solution: **Barrenblatt solution**

$$u(t,x,y) = t^{-\alpha}[(c - k\|y-x\|^2 t^{-2\beta})_+]^{\frac{1}{m-1}},$$

where $\alpha := \frac{d}{d(m-1)+2}$, $\beta := \frac{\alpha}{d}$, $k := \frac{\alpha(m-1)}{2md}$ and $c > 0$ s.th. $\int_{\mathbb{R}^d} u(t,x,y)\,\mathrm{d}y = 1$.

P R O B A B I L I T Y

$\exists$ probability measure $\mathbb{P}_x$ on $C([0,\infty);\mathbb{R}^d)_x$ such that

$$\underbrace{(X(t))^*(\mathbb{P}_x)}_{\text{"push forward"}}(\mathrm{d}y) = u(t,x,y)\,\mathrm{d}y,\ \ t > 0. \text{ (McKean!)}$$

and $\mathbb{P}_x := \underbrace{(X)^*(\mathbb{W}_x)}_{\text{"push forward"}}$, where

$X : C([0,\infty);\mathbb{R}^d)_x \to C([0,\infty);\mathbb{R}^d)_x$ solution of

$$\mathrm{d}X(t) = |u(t,x,X(t))|^{m-1}\mathrm{d}W(t),\ \ t \geq 0, (X(0))^*(\mathbb{P}_x) = \delta_x.$$

$((X(t))_{t\geq 0}, \mathbb{P}_x)_{x\in\mathbb{R}^d}$ "nonlinear Brownian motion"

[Barbu/R. 2020, Ren/R./Wang 2022, Rehmeier/R. 2022]

McKean's vision, PNAS 1966!

GENERAL

Nonlinear Parabolic PDE (more precisely: **nonlinear** Fokker-Planck equation)

$\updownarrow$

nonlinear (time-inhomogeneous) Markov process (described by DDSDE)

1. (Fractional) porous media equation perturbed by a transport term

1.1 Recap: "Local" case

Nonlinear Fokker–Planck equation (**distributional solutions**)

$$\frac{\partial}{\partial t}\varrho(t,x) - \Delta_x(\beta(\varrho(t,x))) + \mathrm{div}_x(D(x)b(\varrho(t,x))\varrho(t,x)) = 0,$$

$$\forall (t,x) \in (0,\infty) \times \mathbb{R}^d, \qquad \text{(FPE)}$$

$$\varrho(0,\cdot) = \mu_0 \in \mathcal{M}_b(\mathbb{R}^d)$$

where $\mathcal{M}_b(\mathbb{R}^d)$ = all signed Borel measures on $\mathbb{R}^d$ of bounded variation

Our approach: solve this first!

(nonlinear) superposition principle [Barbu/R.: AOP 2020] $\Downarrow$ $\Uparrow$ Îto (or Dynkin formula)

McKean-Vlasov SDE (probabilistically weak sense)

$$dX(t) = D(X(t))b(\varrho(t,X(t)))dt + \left(\frac{2\beta(\varrho(t,X(t))}{\varrho(t,X(t))}\right)^{\frac{1}{2}} dW(t), \ t \geq 0,$$

$$\mathbb{P} \circ X(0)^{-1} = \mu_0, \ \mathbb{P} \circ X(t)^{-1}(dx) = \varrho(t,x)dx, \ t > 0. \qquad \text{(MVSDE)}$$

Definition

A function $\varrho : [0,\infty) \to \mathcal{M}_b$ is said to be a **distributional solution** to (FPE) if

$$\varrho, \beta(\varrho) \in L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^d),$$

$$\int_0^\infty \int_{\mathbb{R}^d} \varrho(t,x)\left[\varphi_t(t,x) + b(\varrho(t,x))D(x)\cdot\nabla\varphi(t,x)\right] + \beta(\varrho(t,x))\Delta\varphi(t,x)\,dt\,dx \qquad \text{(wFPE)}$$

$$+\int \varphi(0,x)\mu_0(dx) = 0, \forall\varphi\in C_0^\infty([0,\infty)\times\mathbb{R}^d).$$

1.2 Today: "Nonlocal" (=fractional in space) case

Nonlinear fractional Fokker–Planck equation (**distributional solutions**)

$$\frac{\partial}{\partial t}\varrho(t,x) + (-\boldsymbol{\Delta}_x)^s(\beta(\varrho(t,x))) + \operatorname{div}_x(D(x)b(\varrho(t,x))\varrho(t,x)) = 0,$$

$$\forall (t,x) \in (0,\infty)\times\mathbb{R}^d,\ s \in \left(\frac{1}{2},1\right), \qquad (\mathrm{FPE}_s)$$

$$\varrho(0,\cdot) = \mu_0 \in \mathcal{M}_b(\mathbb{R}^d)$$

Our approach: solve this first!

(nonlinear) **nonlocal** superposition principle [R./Xie/Zhang: PTRF 2020] $\Downarrow$ $\Uparrow$ Îto (or Dynkin formula)

McKean-Vlasov SDE **with multiplicative Lévy noise** (probabilistically weak sense)

$$dX(t) = D(X(t))b(\varrho(t,X(t)))dt + \left(\frac{2\beta(\varrho(t,X(t\text{-}))}{\varrho(t,X(t\text{-}))}\right)^{\frac{1}{2s}} \mathbf{dL(t)},\ t \geq 0,$$

$$\mathbb{P}\circ X(0)^{-1} = \mu_0,\ \mathbb{P}\circ X(t)^{-1}(dx) = \varrho(t,x)dx,\ t > 0. \quad (\mathrm{MVSDE}_s)$$

Definition

A function $\varrho : [0,\infty) \to \mathcal{M}_b$ is said to be a **distributional solution** to (FPE_s) if

$$\varrho, \beta(\varrho) \in L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^d),$$

$$\int_0^\infty \int_{\mathbb{R}^d} \varrho(t,x)\,[\varphi_t(t,x) + b(\varrho(t,x))D(x)\cdot\nabla\varphi(t,x)] + \beta(\varrho(t,x)) - (-\boldsymbol{\Delta})^s\varphi(t,x)dt\,dx \tag{wFPE$_s$}$$

$$+\int \varphi(0,x)\mu_0(dx) = 0, \forall\varphi\in C_0^\infty([0,\infty)\times\mathbb{R}^d).$$

1.3 Next time: "Nonlocal"(=Bernstein) case

Let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a **Bernstein function.**

Nonlinear fractional Fokker–Planck equation (**distributional solutions**)

$$\frac{\partial}{\partial t}\varrho(t,x) + \boldsymbol{\Psi}(-\boldsymbol{\Delta}_x)(\beta(\varrho(t,x))) + \mathrm{div}_x(D(x)b(\varrho(t,x))\varrho(t,x)) = 0,$$
$$\forall (t,x) \in (0,\infty) \times \mathbb{R}^d, \qquad (\mathrm{FPE}_\psi)$$
$$\varrho(0,\cdot) = \mu_0 \in \mathcal{M}_b(\mathbb{R}^d)$$

Our approach: solve this first!

(nonlinear) **nonlocal** superposition principle [R./Xie/Zhang: PTRF 2020] $\Downarrow$ $\Uparrow$ Îto (or Dynkin formula)

McKean-Vlasov SDE **with multiplicative Lévy noise** (probabilistically weak sense)

$\mathbb{P}$ probability measure on $\mathbb{D}([0,\infty);\mathbb{R}^d)$ solving the martingale problem for $(\mathcal{L}_t, C_c^2(\mathbb{R}^d))$ such that

$$\mathbb{P}\circ X(0)^{-1} = \mu_0,\ \mathbb{P}\circ X(t)^{-1}(dx) = \varrho(t,x)dx,\ t>0. \qquad (\mathrm{MVSDE}_\psi)$$

Here

$$\mathcal{L}_t f(x) = b(\varrho(t,x))\, D(x) \,\cdot \nabla f(x) + \frac{\beta(\varrho(t,x))}{\varrho(t,x)}\, p.v. - \int_{\mathbb{R}^d} (f(x+z) - f(x))\nu_{\boldsymbol{\Psi}}(dz)$$

with

$$\nu_{\boldsymbol{\Psi}}(dz) = \int_0^\infty (2t)^{-\frac{d}{2}}\, e^{-\frac{|z|^2}{2t}}\, \mu(dt)dz \text{ and } \boldsymbol{\Psi}(r) = \int_0^\infty (1 - e^{-rt})\mu(dt).$$

Definition

A function $\varrho : [0,\infty) \to \mathcal{M}_b$ is said to be a **distributional solution** to (FPE_ψ) if

$$\varrho, \beta(\varrho) \in L^1_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^d),$$

$$\int_0^\infty \int_{\mathbb{R}^d} \varrho(t,x)\, [\varphi_t(t,x) + b(\varrho(t,x))D(x) \cdot \nabla\varphi(t,x)] + \beta(\varrho(t,x)) - \boldsymbol{\Psi}(-\boldsymbol{\Delta})\varphi(t,x) dt\, dx \qquad (\mathrm{wFPE}_\psi)$$

$$+ \int \varphi(0,x)\mu_0(dx) = 0, \forall \varphi \in C_0^\infty([0,\infty) \times \mathbb{R}^d).$$

2. Nonlinear FPE with fractional Laplacian: existence of distributional solutions

From now on: Change of notation: $\boldsymbol{u(t,x)}$ instead of $\varrho(t,x)$.

Consider the nonlocal nonlinear Fokker–Planck equation (N^2FPE)

$$\begin{aligned} &u_t + (-\Delta)^s \beta(u) + \mathrm{div}(Db(u)u) = 0 \ \text{ on } (0,\infty)\times\mathbb{R}^d,\\ &u(0,x) = u_0(x),\ x\in\mathbb{R}^d, \end{aligned} \qquad (\mathrm{N^2FPE})$$

where $d \geq 2$ and $(-\Delta)^s$, $\frac{1}{2} < s < 1$, is the fractional Laplace operator defined as follows. Let $S' := S'(\mathbb{R}^d)$ be the dual of the Schwartz test function space $S := S(\mathbb{R}^d)$. Define

$$D_s := \{u \in S';\ \mathcal{F}(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^d),\ |\xi|^{2s}\mathcal{F}(u) \in S'\}\ (\supset L^1(\mathbb{R}^d))$$

and

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi),\ \xi\in\mathbb{R}^d,\ u\in D_s,$$

where $\mathcal{F}$ stands for the Fourier transform in $\mathbb{R}^d$, that is,

$$\mathcal{F}(u)(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{ix\cdot\xi}u(x)dx,\ \xi\in\mathbb{R}^d,\ u\in L^1(\mathbb{R}^d).$$

($\mathcal{F}$ extends from S' to itself.)

(N^2FPE) is used for modelling the dynamics of anomalous diffusion of particles in disordered media.

Hypotheses

(i) $\beta \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R}),\ \beta(0) = 0,\ \beta'(r) > 0,\ \forall\, r \neq 0.$

(ii) $D \in C^1(\mathbb{R}^d; \mathbb{R}^d),\ \mathrm{div}\, D \in L^2_{\mathrm{loc}}(\mathbb{R}^d).$

(iii) $b \in C_b(\mathbb{R}).$

(iv) $(\mathrm{div}\, D)^- \in L^\infty,\ \ b \geq 0.$

Define an operator on $L^1(\mathbb{R}^d)$

$$\begin{aligned} A_0(u) \ &:= (-\Delta)^s \beta(u) + \mathrm{div}(Db(u)u),\ \ u \in D(A_0), \\ D(A_0) \ &:= \left\{ u \in L^1(\mathbb{R}^d); (-\Delta)^s \beta(u) + \mathrm{div}(Db(u)u) \in L^1(\mathbb{R}^d) \right\}, \end{aligned}$$

where div is taken in the sense of Schwartz distributions on $\mathbb{R}^d$.

The following lemma is crucial.

Lemma 1

Assume that $\frac{1}{2} < s < 1$*. Then, under Hypotheses* (i)–(iv) *there is* $\lambda_0 > 0$ *and a family of operators* $\{J_\lambda : L^1 \to L^1; \lambda > 0)\}$ **("nonlinear resolvent")**, *which for all* $\lambda \in (0, \lambda_0)$ *satisfies*

$$(I + \lambda A_0)(J_\lambda(f)) = f, \ \forall f \in L^1,$$

$$|J_\lambda(f_1) - J_\lambda(f_2)|_1 \leq |f_1 - f_2|_1, \ \forall f_1, f_2 \in L^1,$$

$$J_{\lambda_2}(f) = J_{\lambda_1}\left(\frac{\lambda_1}{\lambda_2} f + \left(1 - \frac{\lambda_1}{\lambda_2}\right) J_{\lambda_2}(f)\right), \ \forall f \in L^1, \ \lambda_1, \lambda_2 > 0,$$

$$\int_{\mathbb{R}^d} J_\lambda(f) dx = \int_{\mathbb{R}^d} f \, dx, \ \forall f \in L^1,$$

$$J_\lambda(f) \geq 0, \ \ \textit{a.e. on } \mathbb{R}^d, \ \textit{if } f \geq 0, \ \textit{a.e. on } \mathbb{R}^d,$$

$$|J_\lambda(f)|_\infty \leq (1 + ||D| + (\mathrm{div}\, D)^-|_\infty^{\frac{1}{2}})|f|_\infty, \ \forall f \in L^1 \cap L^\infty,$$

$$\beta(J_\lambda(f)) \in H^s \cap L^1 \cap L^\infty, \ \forall f \in L^1 \cap L^\infty.$$

Proof of Lemma 1

Hard! About 11 pages. Idea: Prove existence of a solution $y = y_\lambda \in D(A_0)$ *to the equation*

$$y + \lambda A_0(y) = f \text{ in } S',$$

for $f \in L^1$ *by first considering the approximating equation*

$$y + \lambda(\varepsilon I - \Delta)^s(\beta_\varepsilon(y)) + \lambda \operatorname{div}(D_\varepsilon b_\varepsilon(y)y) = f \text{ in } S',$$

where $\varepsilon \in (0,1]$ *and for* $r \in \mathbb{R}$, $\beta_\varepsilon(r) := \beta(r) + \varepsilon r$ *and*

$$D_\varepsilon := \eta_\varepsilon D,\ \eta_\varepsilon \in C_0^1(\mathbb{R}^d),\ 0 \le \eta_\varepsilon \le 1,\ |\nabla \eta_\varepsilon| \le 1,\ \eta_\varepsilon(x) = 1 \text{ if } |x| < \frac{1}{\varepsilon},$$
$$b_\varepsilon(r) \equiv \frac{(b * \varphi_\varepsilon)(r)}{1 + \varepsilon|r|},$$

where $\varphi_\varepsilon(r) = \frac{1}{\varepsilon}\,\varphi\left(\frac{r}{\varepsilon}\right)$ *is a standard mollifier.*

Then let $\varepsilon \to 0$. □

Now define

$$\begin{aligned} D(A) &:= J_\lambda(L^1)\ (\subset D(A_0)),\\ A(y) &:= A_0(y),\ y \in D(A). \end{aligned}$$

Again it is easy to see that $J_\lambda(L^1)$ is independent of $\lambda \in (0, \lambda_0)$ and that

$$J_\lambda = (I + \lambda A)^{-1},\ \lambda \in (0, \lambda_0). \quad (\text{"Nonlinear resolvent"})$$

Then we have

Lemma 2

Under Hypotheses (i)–(iv)*, the operator* $(A, D(A))$ *defined above is m-accretive in* L^1 *and* $(I + \lambda A)^{-1} = J_\lambda$*,* $\lambda \in (0, \lambda_0)$*. Moreover, if* $\beta \in C^\infty(\mathbb{R})$*, then* $\overline{D(A)} = L^1$*.*

Here, $\overline{D(A)}$ is the closure of $D(A)$ in L^1.

Then by the Crandall-Liggett theorem for $u_0 \in \overline{D(A)},\ t \geq 0$

$$S(t)u_0 := \lim_{n\to 0}(I + \frac{t}{n}A)^{-n}u_0 \,(= ''e^{-tA}u_0'') \qquad \text{("Euler formula")}$$

converges in L^1 and we have the following:

Theorem I

Assume that Hypotheses (i)–(iv) *hold. Then $S(t) : L^1 \to L^1$, $t \geq 0$, is a C_0-semigroup of contractions such that for each $u_0 \in \overline{D(A)}$, $(= L^1$, if $\beta \in C^\infty(\mathbb{R}))$, $u(t, u_0) := S(t)u_0$ is* **a distributional solution** *to* ($\mathrm{N^2FPE}$)*. Moreover, if $u_0 \geq 0$, a.e. on $\mathbb{R}^d$,*

$$u(t, u_0) \geq 0, \quad \text{a.e. on } \mathbb{R}^d,\ \forall\, t \geq 0,$$

and

$$\int_{\mathbb{R}^d} u(t, u_0)(x)dx = \int_{\mathbb{R}^d} u_0(x)dx, \quad \forall\, t \geq 0.$$

Finally, if $u_0 \in L^1 \cap L^\infty$, then all above assertions remain true, if we drop the assumption $\beta \in \mathrm{Lip}(\mathbb{R})$ from Hypothesis (i)*, and additionally we have that* $\boldsymbol{u \in L^\infty((0, T) \times \mathbb{R}^d), T > 0}$.

3. Nonlinear FPE with fractional Laplacian: uniqueness of distributional solutions

Assume $s \in (\frac{1}{2}, 1)$ and

Hypotheses

(j) $\beta \in C^1(\mathbb{R}),\ \beta'(r) > 0,\ \forall\, r \in \mathbb{R},\ \beta(0) = 0.$

(jj) $D \in L^\infty(\mathbb{R}^d; \mathbb{R}^d).$

(jjj) $b \in C^1(\mathbb{R}).$

Theorem II

Let $s \in \left(\frac{1}{2}, 1\right)$, $T > 0$, and let $y_1, y_2 \in L^\infty((0,T)\times\mathbb{R}^d)$ be two distributional solutions to (N^2FPE) *on $(0,T) \times \mathbb{R}^d$ such that $y_1 - y_2 \in L^1((0,T) \times \mathbb{R}^d) \cap L^\infty(0,T; L^2)$ and*

$$\lim_{t\to 0} \operatorname*{ess\,sup}_{s\in(0,t)} |(y_1(s) - y_2(s), \varphi)_2| = 0,\ \forall \varphi \in C_0^\infty(\mathbb{R}^d). \tag{IC}$$

Then $y_1 \equiv y_2$. If $D \equiv 0$, then Hypothesis (j) *can be relaxed to*

(j)′ $\beta \in C^1(\mathbb{R}),\ \beta'(r) \geq 0,\ \forall\, r \in \mathbb{R},\ \beta(0) = 0.$

Theorem III ("Linearized uniqueness.")

Under assumptions of Theorem II, let $T > 0$, $u \in L^\infty((0,T) \times \mathbb{R}^d)$ *and let* $y_1, y_2 \in L^\infty((0,T) \times \mathbb{R}^d)$ *with* $y_1 - y_2 \in L^1((0,T) \times \mathbb{R}^d) \cap L^\infty(0,T; L^2)$ *be two* **distributional solutions** *to the equation*

$$y_t + (-\Delta)^s \left(\frac{\beta(u)}{u} y \right) + \operatorname{div}(yDb(u)) = 0 \text{ on } \mathcal{D}'((0,T) \times \mathbb{R}^d),$$

$$y(0) = u_0,$$

where u_0 *is a measure of finite variation on* $\mathbb{R}^d$ *and* $\frac{\beta(0)}{0} := \beta'(0)$. *If* (IC) *holds, then* $y_1 \equiv y_2$.

4. Applications to McKean–Vlasov SDEs of Nemytskii–type with Levy–Noise
4.1 Existence

Theorem IV

Assume that Hypotheses (i)–(iv) *hold and let* $u_0 \in L^1$. *Assume that* $u_0 \in \overline{D(A)}$ $(= L^1$, *if* $\beta \in C^\infty(\mathbb{R}))$ *and let* u *be the solution of* $(\mathrm{N}^2\mathrm{FPE})$ *from Theorem I. Then, there exist a stochastic basis* $\mathbb{B} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *and a* **d-dimensional isotropic $2s$-stable process L with Lévy measure $\frac{dz}{|z|^{d+2s}}$** *as well as an* $(\mathcal{F}_t)$*-adapted càdlàg process* (X_t) *on* Ω *such that, for*

$$\mathcal{L}_{X_t}(x) := \frac{d(\mathbb{P} \circ X_t^{-1})}{dx}(x), \ \ t \geq 0,$$

we have

$$\begin{aligned} dX_t &= D(X_t)b(\mathcal{L}_{X_t}(X_t))dx + \left(\frac{2\beta(\mathcal{L}_{X_t}(X_{t-}))}{\mathcal{L}_{X_t}(X_{t-})}\right)^{\frac{1}{2s}} dL_t, \\ \mathcal{L}_{X_0} &= u_0. \end{aligned} \qquad (\mathrm{MVSDE}_s)$$

Furthermore,

$$\mathcal{L}_{X_t} = u(t,\cdot), \ \ t \geq 0,$$

in particular, $((t,x) \mapsto \mathcal{L}_{X_t}(x)) \in L^\infty([0,T] \times \mathbb{R}^d)$ *for every* $T > 0$.

Proof. By the well known formula that

$$(-\Delta)^s f(x) = -c_{d,s}\mathrm{P.V.} - \int_{\mathbb{R}^d} (f(x+z) - f(x))\,\frac{dz}{|z|^{d+2s}}$$

with $c_{d,s} \in (0,\infty)$, and since, as an easy calculation shows,

$$\int_A \frac{\beta(u(t,x))}{u(t,x)}\,\frac{dz}{|z|^{d+2s}} = \int_{\mathbb{R}^d} \mathbf{1}_A\left(\left(\frac{\beta(u(t,x))}{u(t,x)}\right)^{\frac{1}{2s}} z\right)\frac{dz}{|z|^{d+2s}},$$

$$A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$

we have

$$\frac{\beta(u(t,x))}{u(t,x)}\,(-\Delta)^s f(x)$$

$$= -c_{d,s}\mathrm{P.V.} \int_{\mathbb{R}^d}\left(f\left(x + \left(\frac{\beta(u(t,x))}{u(t,x)}\right)^{\frac{1}{2s}} z\right) - f(x)\right)\frac{dz}{|z|^{d+2s}}.$$

Hence by Hypotheses (i)-(iv) the (nonlocal!) superposition principle proved in [R./Xie/Zhang, PTRF 2020] applies to show that there exists a stochastic basis $\mathbb{B}$ and $(X_t)_{t\geq 0}$ as in the assertion of the theorem, as well as a Poisson random measure N on $\mathbb{R}^d \times [0,\infty)$ with intensity $|z|^{-d-2s}dz\,dt$ on the stochastic basis $\mathbb{B}$ such that for

$$L_t := \int_0^t \int_{|z|\leq 1} z\widetilde{N}(dz\,ds) + \int_0^t \int_{|z|>1} z\,N(dz\,ds),$$

with

$$\widetilde{N}(dz\,dt) := N(dz\,dt) - |z|^{-d-2s}dz\,dt$$

(MVSDE$_s$) holds. □

4.2 Uniqueness

Theorem V

Assume that Hypotheses (j)–(jjj)*, resp.* (j)′, (jj), (jjj) *if* $D \equiv 0$*, hold and let* $T > 0$*. Let* (X_t) *and* $(\widetilde{X}_t)$ *be two càdlàg processes on two (possibly different) stochastic bases* $\mathbb{B}, \widetilde{\mathbb{B}}$ *that are weak solutions to (MVSDE) with (possibly different)* L *and* $\widetilde{L}$*. Assume that*

$$((t,x) \mapsto \mathcal{L}_{X_t}(x)), \left((t,x) \mapsto \mathcal{L}_{\widetilde{X}_t}(x)\right) \in L^\infty((0,T) \times \mathbb{R}^d).$$

Then X *and* $\widetilde{X}$ *have the same laws, i.e.,*

$$\mathbb{P} \circ X^{-1} = \widetilde{\mathbb{P}} \circ \widetilde{X}^{-1}.$$

Proof. Clearly, by Dynkin's formula both

$$\mu_t(dx) := \mathcal{L}_{X_t}(x)dx \text{ and } \widetilde{\mu}_t(dx) := \mathcal{L}_{\widetilde{X}_t}(x)dx$$

solve (N^{2}FPE) with the same initial condition $u_0(dx) := u_0(x)dx$, hence satisfy (IC) with $y_1(t) := \mathcal{L}_{X_t}$ and $y_2(t) := \mathcal{L}_{\widetilde{X}_t}$. Hence, by Theorem II,

$$\mathcal{L}_{X_t} = \mathcal{L}_{\widetilde{X}_t} \text{ for all } t \geq 0,$$

since $t \mapsto \mathcal{L}_{X_t}(x)dx$ and $t \mapsto \mathcal{L}_{\widetilde{X}_t}(x)dx$ are both narrowly continuous and are probability measures for all $t \geq 0$, so both are in $L^\infty(0,T;L^1 \cap L^\infty) \subset L^\infty(0,T;L^2)$.

Now, consider the linear Fokker–Planck equation

$$\begin{aligned} v_t + (-\Delta)^s \, \frac{\beta(\mathcal{L}_{X_t})}{\mathcal{L}_{X_t}} v + \operatorname{div}(Db(\mathcal{L}_{X_t})v) = 0, \\ v(0,x) = u_0(x), \end{aligned} \qquad \text{(lFPE)}$$

again in the distributional sense. Then, by Theorem III we conclude that $\mathcal{L}_{X_t}$, $t \in [0,T]$, is the unique solution to (lFPE) in $L^\infty(0,T;L^1 \cap L^\infty)$. Again by Dynkin's formula, both $\mathbb{P} \circ X^{-1}$ and $\widetilde{\mathbb{P}} \circ \widetilde{X}^{-1}$ solve the martingale problem with initial condition $u_0(dx) := u_0(x)dx$ for the linear Kolmogorov operator

$$K_{\mathcal{L}_{X_t}} := -\frac{\beta(\mathcal{L}_{X_t})}{\mathcal{L}_{X_t}} (-\Delta)^s + b(\mathcal{L}_{X_t}) D \cdot \nabla.$$

Since the above is true for all $u_0 \in L^1 \cap L^\infty$, and also holds when we consider (N^{2}FPE) and (lFPE) with start in any $s_0 > 0$ instead of zero, it follows by exactly the same arguments as in the proof of Lemma 2.12 in [Trevisan: EJP 2016] that

$$\mathbb{P} \circ X^{-1} = \widetilde{\mathbb{P}} \circ \widetilde{X}^{-1}. \qquad \square$$

4.3 Nonlinear Markov process in the sense of McKean

Let for $s \in [0,\infty)$ and $Z := \{\zeta \equiv \zeta(x)dx \mid \zeta \in L^1 \cap L^\infty,\ \zeta \geq 0,\ |\zeta|_1 = 1\}$

$$\mathbb{P}_{(s,\zeta)} := \mathbb{P} \circ X^{-1}(s,\zeta),$$

where $(X_t(s,\zeta))_{t\geq 0}$ on a stochastic basis $\mathbb{B}$ denotes the solution of (MVSDE$_s$) with initial condition ζ at s. Then by Theorems II, III and V, it follows that $\mathbb{P}_{(s,\zeta)}$, $(s,\zeta) \in [0,\infty) \times Z$, form a nonlinear Markov process in the sense of [McKean: PNAS 1966]. and [Rehmeier/R.:arXiv 2212.12424v2]. For the proof see the latter paper and [Ren/R. Wang: JDE 2022, Corollary 4.6].